Closed-Form Solution of Seventh-Order Navigation System for Unaided Flyout

Jerome J. Krempasky*
Raytheon Electronic Systems, Tucson, Arizona 85734

Introduction

SEVERAL authors have generated time-dependent solutions to various sets of navigation error state differential equations governing the unaided flyout of a guided weapon.[1–5] Developed herein is a closed-form solution to the seventh-order system of horizontal-plane navigation error state equations governing the flyout of a guided weapon before its first navigational update. During the unaided navigation, which is usually preceded by a transfer alignment process, the vertical channel is assumed to be stabilized by barometric or radar altimeter measurements. This solution represents an enhancement to the previously derived results[1–5] in that the appropriate number of error states are analyzed for the navigation problem under consideration, a moving guided weapon is assumed, the exact eigenvalues are extracted from the system $\boldsymbol{F}$ matrix without invoking any approximations, and the appropriate driving terms (the three gyro biases) are treated in the correct manner.

The initial step in the solution process involves extracting the eigenfrequencies of the system dynamics matrix $\boldsymbol{F}$, a process that yields frequencies near the Schuler frequency, a frequency about 15 times smaller than the Schuler frequency (the 24-h variation), and the zero frequency (pole at the origin). The seven system eigenvectors are then derived by solving the eigenvector equation in the standard fashion.[6] Next, the problem is restricted to a weapon flyout in an easterly (or westerly) direction. This restriction is necessary to permit a closed-form analytic solution when later the initial conditions are incorporated into the problem. Because flyout in an easterly direction results in the largest crosstrack error, this restriction encompasses the worst-case scenario. The next step involves determining the particular solution to the system dynamics equation (i.e., the solution that results from the three gyro bias inputs), by employing the method of undetermined coefficients. The total solution, at this point, is a sum of the homogeneous solution, which is a linear combination of the seven linearly independent solutions associated with the eigenvalues, and the particular solution. The final step in the solution process is the incorporation of initial conditions, which permits determination of the seven arbitrary coefficients present in the linearly independent solutions constituting the homogeneous solution.

In terms of providing a practical usage of the equations derived herein, it is demonstrated that the exact solution for the crosstrack error can be utilized to construct an alignment quality indicator (AQI) that can be invoked during the transfer alignment process for the guided weapon. In particular, an AQI should have the ability to predict, at any time during the alignment process, the uncertainty in the crosstrack error that would result if the weapon were to be launched at that time and then fly unaided at a constant velocity and heading for some prescribed period of time. The period of time that the weapon flies unaided is the time to the first navigational update [terrain scene or global positioning system (GPS)]. In calculating the expected crosstrack error at the first navigational update time, the AQI must utilize the current quality of alignment as provided by the appropriate variances found in the weapon Kalman filter covariance matrix.

Differential Equations for Unaided Flyout

The unaided flyout mission considered herein is assumed to occur at constant heading, altitude, and speed with only vertical updates being supplied to the guided weapon Kalman filter to keep the vertical channel stable. The appropriate seven error state differential equations are

$$
\begin{aligned}
&\delta\dot{x} = \delta V_x - V_y\delta\theta_z, \qquad \delta\dot{y} = \delta V_y + V_x\delta\theta_z \\
&\delta\dot{V}_x = 2\Omega_z\delta V_y - G\phi_y + \delta f_x, \qquad \delta\dot{V}_y = -2\Omega_z\delta V_x + G\phi_x + \delta f_y \\
&\dot{\phi}_x = -(\Omega_z/R)\delta x - (1/R)\delta V_y + \Omega_z\phi_y + \Omega_y\delta\theta_z - u_x \\
&\dot{\phi}_y = -(\Omega_z/R)\delta y + (1/R)\delta V_x - \Omega_z\phi_x - \Omega_x\delta\theta_z - u_y \\
&\delta\dot{\theta}_z = -(\omega_x/R)\delta x - (\omega_y/R)\delta y - \omega_y\phi_x + \omega_x\phi_y - u_z
\end{aligned}
\tag{1}
$$

where the craft and Earth rate components are

$$
\begin{aligned}
&\omega_x = \Omega_x - V_y/R, \qquad \Omega_x = \Omega_e\cos L\sin\alpha \\
&\omega_y = \Omega_y + V_x/R, \qquad \Omega_y = \Omega_e\cos L\cos\alpha, \qquad \Omega_z = \Omega_e\sin L
\end{aligned}
\tag{2}
$$

The first two expressions in Eq. (1) govern the horizontal position error components, the second two the horizontal velocity error components, and the final three the tilt angle error components. Here, V_x and V_y are the horizontal components of the Earth-referenced weapon velocity resolved in a wander azimuth local-level coordinate system that has its z axis pointing up and its x axis pointing east, when the wander angle α is zero. In addition, R denotes the radius of the Earth, G the acceleration due to gravity, L the geodetic latitude, δf_x and δf_y the horizontal components of the inertial measurement unit (IMU) accelerometer instrument errors, and u_x, u_y, and u_z the components of the IMU gyro instrument errors resolved in the local-level system.

Neglected in the seven error state differential equations is process noise because its inclusion would preclude an exact solution of the coupled system of equations. The equations are written in the form in which ϕ_z, rather than $\delta\theta_z$, is taken to be zero.[7] If the third and fifth expressions in Eq. (1) are combined, the two terms $-u_x + \Omega_z\delta f_x/G$ will result. For high-quality navigators, it can be shown that $(\Omega_z/G)\delta f_i \leq 0.1u_i$ where i denotes x or y. Hence, the accelerometer errors can be neglected in comparison to the gyro errors for the unaided flyout part of the mission, assuming a high-quality navigator.

Eigenvalues and Eigenvectors of the Seventh-Order System

The seventh-order system of Eq. (1) can be rewritten in matrix form as

$$
\dot{\boldsymbol{x}} = \boldsymbol{F}\boldsymbol{x} + \boldsymbol{u}, \qquad \boldsymbol{x}^T = [\delta x \quad \delta y \quad \delta V_x \quad \delta V_y \quad \phi_x \quad \phi_y \quad \delta\theta_z]
$$
$$
\boldsymbol{u}^T = [0 \quad 0 \quad 0 \quad 0 \quad -u_x \quad -u_y \quad -u_z] \tag{3}
$$

where the system dynamics matrix is given by

$$
\boldsymbol{F} = \begin{bmatrix}
0 & 0 & 1 & 0 & 0 & 0 & -V_y \\
0 & 0 & 0 & 1 & 0 & 0 & V_x \\
0 & 0 & 0 & 2\Omega_z & 0 & -G & 0 \\
0 & 0 & -2\Omega_z & 0 & G & 0 & 0 \\
-\Omega_z/R & 0 & 0 & -1/R & 0 & \Omega_z & \Omega_y \\
0 & -\Omega_z/R & 1/R & 0 & -\Omega_z & 0 & -\Omega_x \\
-\omega_x/R & -\omega_y/R & 0 & 0 & -\omega_y & \omega_x & 0
\end{bmatrix} \tag{4}
$$

To find the solution to the homogeneous equation, $\dot{\boldsymbol{x}} = \boldsymbol{F}\boldsymbol{x}$, it is necessary to find the eigenvalues of $\boldsymbol{F}$. After lengthy algebraic manipulations, the following seven eigenvalues are obtained:

$$
\lambda_0 = 0, \qquad \lambda_{1,2} = \pm j\omega_1, \qquad \lambda_{3,4} = \pm j\omega_2, \qquad \lambda_{5,6} = \pm j\omega_3 \tag{5}
$$

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*Senior Engineering Fellow, Guidance, Navigation, and Control Center, P.O. Box 11337.

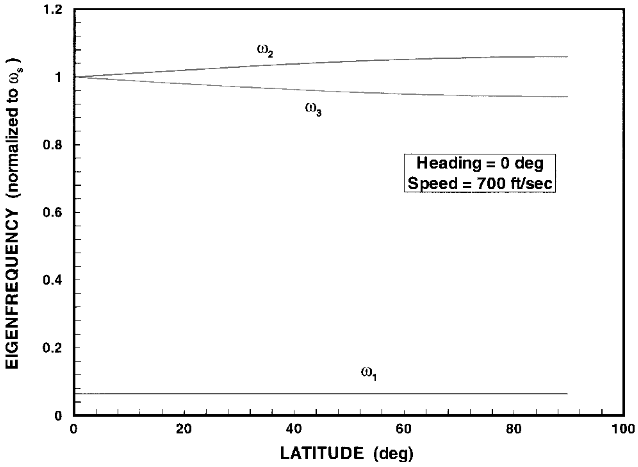

Fig. 1 Eigenfrequencies for unaided flyout.

where

$$\omega_1^2 = -2\sqrt{-a/3}\cos(\phi/3) + p/3, \qquad p = \omega_c^2 + 5\Omega_z^2 + 2\omega_s^2$$

$$\omega_2^2 = -2\sqrt{-a/3}\cos(\phi/3 + 2\pi/3) + p/3$$

$$q = \omega_s^4 + 2\left(\omega_s^2 + 2\Omega_z^2\right)\left(\omega_c^2 + \Omega_z^2\right)$$

$$\omega_3^2 = -2\sqrt{-a/3}\cos(\phi/3 + 4\pi/3) + p/3, \qquad r = \omega_s^4\left(\Omega_z^2 + \omega_c^2\right) \tag{6}$$

$$\cos(\phi) = -b/2\Big/\sqrt{-a^3/27}, \qquad a = (3q - p^2)/3$$

$$b = (2p^3 - 9pq + 27r)/27$$

Here $\omega_c^2 = \omega_x^2 + \omega_y^2$, and ω_s is the Schuler frequency, which is given by $\omega_s^2 = G/R$.

Shown in Fig. 1 are the values assumed by ω_1, ω_2, and ω_3 as a function of latitude for a guided weapon heading of 0° (due north flight). For ease of visualization, these values are normalized to the Schuler frequency, which is 0.00124 rad/s. An examination of Fig. 1 reveals that there is a double root, that is, $\omega_2 = \omega_3 = \omega_s$, at 0° latitude, a degeneracy that occurs for any heading angle. For values of latitude other than 0° the trend seems to be that one eigenfrequency grows to approximately 6% greater than the Schuler frequency, whereas another eigenfrequency grows to approximately 6% less than the Schuler frequency. These eigenfrequencies are quite similar to previously predicted oscillations,[2,3,5] which are basically the Schuler frequency shifted up and down by the Foucault pendulum frequency. The third eigenfrequency, on the other hand, is a factor of 15 or so less than the Schuler frequency and is thus usually referred to as the 24-h oscillation. It is found to be sensitive to the guided weapon speed, for example, it can vary by a factor of 2.5 in going from 0 to 2000 kn.

The eigenvector $\boldsymbol{a}$, which corresponds to the zero eigenvalue, is found by solving the equation $\boldsymbol{Fa} = \boldsymbol{0}$. If the components of $\boldsymbol{a}$ are designated by

$$\boldsymbol{a}^T = [a_1 \quad a_2 \quad a_3 \quad a_4 \quad a_5 \quad a_6 \quad a_7] \tag{7}$$

the solution to $\boldsymbol{Fa} = \boldsymbol{0}$ is found to be

$$a_1 = 1, \qquad a_2 = -\left(G\omega_x + 2V_y\Omega_x^2\right)\Big/D_c, \qquad a_3 = V_y\Omega_z\omega_s^2\Big/D_c$$

$$a_4 = -V_x\Omega_z\omega_s^2\Big/D_c, \qquad a_5 = 2V_y\Omega_z^2\Big/(RD_c)$$

$$a_6 = -2V_x\Omega_z^2\omega_s^2\Big/(GD_c), \qquad a_7 = \Omega_z\omega_s^2\Big/D_c$$

$$D_c = G\omega_y - 2V_x\Omega_z^2 \tag{8}$$

If $\boldsymbol{e}^{(i)}$ is the eigenvector corresponding to the eigenvalue $j\omega_i$ (where $i = 1, 2, 3$), it will be complex and expressible as

$$\boldsymbol{e}^{(i)^T} = \left[1 \quad e_{2R}^{(i)} + je_{2I}^{(i)} \quad \ldots \quad e_{7R}^{(i)} + je_{7I}^{(i)}\right] \tag{9}$$

For the eigenfrequency $-j\omega_i$, the eigenvector is clearly the complex conjugate of $\boldsymbol{e}^{(i)}$. The solution to the homogeneous equation $\dot{\boldsymbol{x}} = \boldsymbol{Fx}$ can, thus, be written

$$\boldsymbol{x}_h = C_1\begin{bmatrix}1\\a_2\\\vdots\\a_7\end{bmatrix} + \sum_{i=1}^{3}\left\{C_{1+i}\begin{bmatrix}\cos\omega_i t\\ e_{2R}^{(i)}\cos\omega_i t - e_{2I}^{(i)}\sin\omega_i t\\ \vdots\\ e_{7R}^{(i)}\cos\omega_i t - e_{7I}^{(i)}\sin\omega_i t\end{bmatrix}\right.$$

$$\left. + C_{4+i}\begin{bmatrix}\sin\omega_i t\\ e_{2R}^{(i)}\sin\omega_i t + e_{2I}^{(i)}\cos\omega_i t\\ \vdots\\ e_{7R}^{(i)}\sin\omega_i t + e_{7I}^{(i)}\cos\omega_i t\end{bmatrix}\right\} \tag{10}$$

where the seven C_j are arbitrary constants to be determined by the initial conditions.

If attention is now restricted to flying out in either the east or west direction, and the wander angle is taken to be zero, then V_y, Ω_x, and, consequently, ω_x, become zero. Note that the restriction to flyout direction encompasses the worst case (flying east) in terms of realizing the largest crosstrack error. In this case, the components of $\boldsymbol{e}^{(i)}$ simplify to

$$e_1^{(i)} = 1, \qquad e_2^{(i)} = je_{2I}^{(i)} = j\left[\frac{\omega_i\left(\omega_s^2 - 2\Omega_z^2 - \omega_i^2\right)}{\Omega_z\omega_s^2} - \frac{3\omega_i e_{4R}^{(i)}}{\omega_s^2}\right]$$

$$e_3^{(i)} = je_{3I}^{(i)} = j\omega_i$$

$$e_4^{(i)} = \frac{V_x\Omega_z^2\omega_s^4 - 3V_x\omega_i^2\omega_s^2\Omega_z^2 + G\omega_i^2\Omega_y\left(\omega_s^2 - 2\Omega_z^2 - \omega_i^2\right)}{\Omega_z\left[3G\omega_i^2\Omega_y + \omega_s^2\left(2V_x\Omega_z^2 + V_x\omega_i^2 - G\omega_y\right)\right]}$$

$$e_{4R}^{(i)} = e_4^{(i)}, \qquad e_5^{(i)} = je_{5I}^{(i)} = j\frac{\omega_i}{G}\left(e_{4R}^{(i)} + 2\Omega_z\right)$$

$$e_6^{(i)} = e_{6R}^{(i)} = \frac{1}{G}\left(2\Omega_z e_{4R}^{(i)} + \omega_i^2\right)$$

$$e_7^{(i)} = e_{7R}^{(i)} = -\frac{1}{V_x}\left(\omega_i e_{2I}^{(i)} + e_{4R}^{(i)}\right) \tag{11}$$

From an examination of Eq. (8), it is seen that the components of $\boldsymbol{a}$ also simplify in this case, specifically, the components a_2, a_3, and a_5 are now precisely zero.

Development of the Particular Solution

Because estimates of the gyro instrument errors are not improved on by the weapon Kalman filter during the flyout mission under consideration, they can be treated as random constants after launch. Hence, the system input vector can be regarded as a constant input to the system dynamics equation. With this in mind, the form of the particular solution to Eq. (3) is

$$\boldsymbol{x}_p = \boldsymbol{b}t + \boldsymbol{v} \tag{12}$$

where $\boldsymbol{b}$ and $\boldsymbol{v}$ denote constant vectors. The term proportional to time on the right-hand side is required because of the constant term $C_1\boldsymbol{a}$ that is present in the homogeneous solution. On substituting Eq. (12) into Eq. (3), the result $\boldsymbol{b} = \boldsymbol{Fb}t + \boldsymbol{Fv} + \boldsymbol{u}$ is obtained. To determine the coefficients of $\boldsymbol{b}$ and $\boldsymbol{v}$ in this method of undetermined coefficients, one must equate the terms proportional to time and, subsequently, the constant terms on both sides, thus yielding

$$\boldsymbol{Fb} = \boldsymbol{0}, \qquad \boldsymbol{b} = \boldsymbol{Fv} + \boldsymbol{u} \tag{13}$$

The first expression appearing here indicates that $\boldsymbol{b}$ is an eigenvector of $\boldsymbol{F}$ corresponding to the eigenvalue zero. Therefore, because $\boldsymbol{a}$ has already been established as the eigenvector of $\boldsymbol{F}$ corresponding to the eigenvalue zero, $\boldsymbol{b}$ is simply proportional to $\boldsymbol{a}$, that is, $\boldsymbol{b} = c\boldsymbol{a}$, where c is as yet an undetermined constant. The second expression in Eq. (13) thus becomes $c\boldsymbol{a} = \boldsymbol{F}\boldsymbol{v} + \boldsymbol{u}$. This equation can be solved for c and $\boldsymbol{v}$, where $\boldsymbol{v}^T = [0\ \ v_2\ \ v_3\ \ v_4\ \ v_5\ \ v_6\ \ v_7]$, if v_1 is taken to be zero, the result being

$$v_2 = D_{2y}u_y + D_{2z}u_z, \qquad v_3 = D_{3y}u_y + D_{3z}u_z, \qquad v_4 = D_{4x}u_x$$
$$v_5 = D_{5y}u_y + D_{5z}u_z, \qquad v_6 = D_{6x}u_x$$
$$v_7 = D_{7x}u_x, \qquad c = v_3 \tag{14}$$

As is to be expected, the components of $\boldsymbol{v}$ are linear combinations of the gyro instrument error inputs. The various coefficients D_{ij} appearing here are defined as follows:

$$D_{2y} = \frac{\Omega_z\left(V_x\omega_y\omega_s^2 - 2G\omega_y^2 + 4V_x\omega_y\Omega_z^2 - G\omega_s^2\right)}{\omega_s^4\left(\Omega_z^2 + \omega_y^2\right)}$$
$$D_{3y} = \frac{\omega_y\left(G\omega_y - 2V_x\Omega_z^2\right)}{\omega_s^2\left(\Omega_z^2 + \omega_y^2\right)}$$
$$D_{2z} = \frac{-G\omega_y\omega_s^2 - V_x\Omega_z^2\omega_s^2 + 2G\omega_y\Omega_z^2 - 4V_x\Omega_z^4}{\omega_s^4\left(\Omega_z^2 + \omega_y^2\right)}$$
$$D_{3z} = -\frac{\Omega_z}{\omega_y}D_{3y}, \qquad D_{4x} = \frac{GV_x}{2V_x\Omega_z^2 - G\omega_y}$$
$$D_{5y} = \frac{(a_4 + 2\Omega_z)D_{3y}}{G}, \qquad D_{5z} = \frac{(a_4 + 2\Omega_z)D_{3z}}{G}$$
$$D_{6x} = \frac{2V_x\Omega_z}{2V_x\Omega_z^2 - G\omega_y}, \qquad D_{7x} = -\frac{G}{2V_x\Omega_z^2 - G\omega_y} \tag{15}$$

Incorporation of Initial Conditions into Solution

The general solution to the system dynamics equation (3) is a sum of the homogeneous and particular solutions, that is, $\boldsymbol{x} = \boldsymbol{x}_h + \boldsymbol{x}_p$. If time is set equal to zero in Eqs. (10) and (12), and the result is then set equal to the state vector $\boldsymbol{x}$ evaluated at $t = 0$, the following matrix equation is obtained:

$$\left\{\begin{matrix} \delta x(0) \\ \delta y(0) - D_{2y}u_y - D_{2z}u_z \\ \delta V_x(0) - D_{3y}u_{3y} - D_{3z}u_z \\ \delta V_y(0) - D_{4x}u_x \\ R[\phi_x(0) - D_{5y}u_y - D_{5z}u_z] \\ R[\phi_y(0) - D_{6x}u_x] \\ R[\delta\theta_z(0) - D_{7x}u_x] \end{matrix}\right\} = \boldsymbol{T}\begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \\ C_5 \\ C_6 \\ C_7 \end{bmatrix} \tag{16}$$

The matrix $\boldsymbol{T}$ contains many zeros as a result of the worst-case assumption of an easterly flyout. Hence, it can be inverted in a closed-form manner, after which Eq. (16) can be solved to yield

$$C_1 = U_{11}\delta x(0) + U_{16}R[\phi_y(0) - D_{6x}u_x] + U_{17}R[\delta\theta_z(0) - D_{7x}u_x]$$
$$C_2 = U_{21}\delta x(0) + U_{26}R[\phi_y(0) - D_{6x}u_x] + U_{27}R[\delta\theta_z(0) - D_{7x}u_x]$$
$$C_3 = U_{31}\delta x(0) + U_{34}[\delta V_y(0) - D_{4x}u_x] + U_{36}R[\phi_y(0) - D_{6x}u_x] + U_{37}R[\delta\theta_z(0) - D_{7x}u_x]$$
$$C_4 = U_{41}\delta x(0) + U_{44}[\delta V_y(0) - D_{4x}u_x] + U_{46}R[\phi_y(0) - D_{6x}u_x] + U_{47}R[\delta\theta_z(0) - D_{7x}u_x]$$
$$C_5 = U_{52}[\delta y(0) - D_{2y}u_y - D_{2z}u_z] + U_{55}R[\phi_x(0) - D_{5y}u_y - D_{5z}u_z]$$
$$C_6 = U_{62}[\delta y(0) - D_{2y}u_y - D_{2z}u_z] + U_{63}[\delta V_x(0) - D_{3y}u_y - D_{3z}u_z] + U_{65}R[\phi_x(0) - D_{5y}u_y - D_{5z}u_z]$$
$$C_7 = U_{72}[\delta y(0) - D_{2y}u_y - D_{2z}u_z] + U_{73}[\delta V_x(0) - D_{3y}u_y - D_{3z}u_z] + U_{75}R[\phi_x(0) - D_{5y}u_y - D_{5z}u_z] \tag{17}$$

The as yet undefined parameters appearing here are

$$U_{11} = \left(e_{6R}^{(1)}e_{7R}^{(2)} - e_{6R}^{(2)}e_{7R}^{(1)}\right)/d_1, \qquad U_{16} = \left(e_{7R}^{(1)} - e_{7R}^{(2)}\right)/Rd_1$$
$$U_{17} = \left(e_{6R}^{(2)} - e_{6R}^{(1)}\right)/Rd_1, \qquad U_{21} = \left(a_7e_{6R}^{(2)} - a_6e_{7R}^{(2)}\right)/d_1$$
$$U_{26} = \left(e_{7R}^{(2)} - a_7\right)/Rd_1, \qquad U_{27} = \left(a_6 - e_{6R}^{(2)}\right)/Rd_1$$
$$U_{31} = \left[a_4\left(e_{6R}^{(1)}e_{7R}^{(3)} - e_{7R}^{(1)}e_{6R}^{(3)}\right) - e_{4R}^{(1)}\left(a_6e_{7R}^{(3)} - a_7e_{6R}^{(3)}\right) + e_{4R}^{(3)}\left(a_6e_{7R}^{(1)} - a_7e_{6R}^{(1)}\right)\right]/d_2$$
$$U_{34} = \left[-\left(e_{6R}^{(1)}e_{7R}^{(3)} - e_{7R}^{(1)}e_{6R}^{(3)}\right) + \left(a_6e_{7R}^{(3)} - a_7e_{6R}^{(3)}\right) - \left(a_6e_{7R}^{(1)} - a_7e_{6R}^{(1)}\right)\right]/d_2$$
$$U_{36} = \left[\left(e_{4R}^{(1)}e_{7R}^{(3)} - e_{7R}^{(1)}e_{4R}^{(3)}\right) - \left(a_4e_{7R}^{(3)} - a_7e_{4R}^{(3)}\right) + \left(a_4e_{7R}^{(1)} - a_7e_{4R}^{(1)}\right)\right]/Rd_2$$
$$U_{37} = \left[-\left(e_{4R}^{(1)}e_{6R}^{(3)} - e_{6R}^{(1)}e_{4R}^{(3)}\right) + \left(a_4e_{6R}^{(3)} - a_6e_{4R}^{(3)}\right) - \left(a_4e_{6R}^{(1)} - a_6e_{4R}^{(1)}\right)\right]/Rd_2$$
$$U_{41} = \left[-a_4\left(e_{6R}^{(1)}e_{7R}^{(2)} - e_{7R}^{(1)}e_{6R}^{(2)}\right) + e_{4R}^{(1)}\left(a_6e_{7R}^{(2)} - a_7e_{6R}^{(2)}\right) - e_{4R}^{(2)}\left(a_6e_{7R}^{(1)} - a_7e_{6R}^{(1)}\right)\right]/d_2$$
$$U_{44} = \left[\left(e_{6R}^{(1)}e_{7R}^{(2)} - e_{7R}^{(1)}e_{6R}^{(2)}\right) - \left(a_6e_{7R}^{(2)} - a_7e_{6R}^{(2)}\right) + \left(a_6e_{7R}^{(1)} - a_7e_{6R}^{(1)}\right)\right]/d_2$$
$$U_{46} = \left[-\left(e_{4R}^{(1)}e_{7R}^{(2)} - e_{7R}^{(1)}e_{4R}^{(2)}\right) + \left(a_4e_{7R}^{(2)} - a_7e_{4R}^{(2)}\right) - \left(a_4e_{7R}^{(1)} - a_7e_{4R}^{(1)}\right)\right]/Rd_2$$
$$U_{47} = \left[\left(e_{4R}^{(1)}e_{6R}^{(2)} - e_{6R}^{(1)}e_{4R}^{(2)}\right) - \left(a_4e_{6R}^{(2)} - a_6e_{4R}^{(2)}\right) + \left(a_4e_{6R}^{(1)} - a_6e_{4R}^{(1)}\right)\right]/Rd_2$$
$$U_{52} = e_{5I}^{(2)}/\left(e_{2I}^{(1)}e_{5I}^{(2)} - e_{5I}^{(1)}e_{2I}^{(2)}\right)$$
$$U_{55} = -e_{2I}^{(2)}/R\left(e_{2I}^{(1)}e_{5I}^{(2)} - e_{5I}^{(1)}e_{2I}^{(2)}\right)$$
$$U_{62} = \left(\omega_3e_{5I}^{(1)} - \omega_1e_{5I}^{(3)}\right)/d_3, \qquad U_{63} = \left(e_{2I}^{(1)}e_{5I}^{(3)} - e_{5I}^{(1)}e_{2I}^{(3)}\right)/d_3$$
$$U_{65} = \left(\omega_1e_{2I}^{(3)} - \omega_3e_{2I}^{(1)}\right)/Rd_3, \qquad U_{72} = \left(\omega_1e_{5I}^{(2)} - \omega_2e_{5I}^{(1)}\right)/d_3$$
$$U_{73} = \left(e_{5I}^{(1)}e_{2I}^{(2)} - e_{2I}^{(1)}e_{5I}^{(2)}\right)/d_3, \qquad U_{75} = \left(\omega_2e_{2I}^{(1)} - \omega_1e_{2I}^{(2)}\right)/Rd_3 \tag{18}$$

The denominators d_i are defined as

$$d_1 = e_{6R}^{(1)}e_{7R}^{(2)} - e_{6R}^{(2)}e_{7R}^{(1)} + a_7e_{6R}^{(2)} - a_6e_{7R}^{(2)} + a_6e_{7R}^{(1)} - a_7e_{6R}^{(1)}$$
$$\begin{aligned} d_2 = {} & e_{4R}^{(1)}\left(e_{6R}^{(2)}e_{7R}^{(3)} - e_{6R}^{(3)}e_{7R}^{(2)}\right) - e_{4R}^{(2)}\left(e_{6R}^{(1)}e_{7R}^{(3)} - e_{6R}^{(3)}e_{7R}^{(1)}\right) \\ & + e_{4R}^{(3)}\left(e_{6R}^{(1)}e_{7R}^{(2)} - e_{6R}^{(2)}e_{7R}^{(1)}\right) - a_4\left(e_{6R}^{(2)}e_{7R}^{(3)} - e_{6R}^{(3)}e_{7R}^{(2)}\right) \\ & + e_{4R}^{(2)}\left(a_6e_{7R}^{(3)} - a_7e_{6R}^{(3)}\right) - e_{4R}^{(3)}\left(a_6e_{7R}^{(2)} - a_7e_{6R}^{(2)}\right) \\ & + a_4\left(e_{6R}^{(1)}e_{7R}^{(3)} - e_{7R}^{(1)}e_{6R}^{(3)}\right) - e_{4R}^{(1)}\left(a_6e_{7R}^{(3)} - a_7e_{6R}^{(3)}\right) \\ & + e_{4R}^{(3)}\left(a_6e_{7R}^{(1)} - a_7e_{6R}^{(1)}\right) - a_4\left(e_{6R}^{(1)}e_{7R}^{(2)} - e_{7R}^{(1)}e_{6R}^{(2)}\right) \\ & + e_{4R}^{(1)}\left(a_6e_{7R}^{(2)} - a_7e_{6R}^{(2)}\right) - e_{4R}^{(2)}\left(a_6e_{7R}^{(1)} - a_7e_{6R}^{(1)}\right) \end{aligned}$$
$$d_3 = e_{2I}^{(1)}\left(\omega_2e_{5I}^{(3)} - \omega_3e_{5I}^{(2)}\right) - e_{2I}^{(2)}\left(\omega_1e_{5I}^{(3)} - \omega_3e_{5I}^{(1)}\right) + e_{2I}^{(3)}\left(\omega_1e_{5I}^{(2)} - \omega_2e_{5I}^{(1)}\right) \tag{19}$$

From Eq. (17), it is seen that the C_j are linear combinations of the initial conditions on the seven error state variables and the gyro instrument errors.

For the unaided flyout mission, the quantity of interest is the crosstrack error expressed as a function of time. Because flyouts with an easterly (or westerly) heading are being considered, the crosstrack error is δy. With the listed values for the C_j, the crosstrack error is

$$
\begin{aligned}
\delta y = {} & \left(-U_{21}e_{21}^{(1)}\sin\omega_1 t - U_{31}e_{21}^{(2)}\sin\omega_2 t - U_{41}e_{21}^{(3)}\sin\omega_3 t\right)\delta x(0) \\
& + \left(U_{52}e_{21}^{(1)}\cos\omega_1 t + U_{62}e_{21}^{(2)}\cos\omega_2 t + U_{72}e_{21}^{(3)}\cos\omega_3 t\right)\delta y(0) \\
& + \left(U_{63}e_{21}^{(2)}\cos\omega_2 t + U_{73}e_{21}^{(3)}\cos\omega_3 t\right)\delta V_x(0) \\
& + \left(-U_{34}e_{21}^{(2)}\sin\omega_2 t - U_{44}e_{21}^{(3)}\sin\omega_3 t\right)\delta V_y(0) \\
& + \left(U_{55}e_{21}^{(1)}\cos\omega_1 t + U_{65}e_{21}^{(2)}\cos\omega_2 t + U_{75}e_{21}^{(3)}\cos\omega_3 t\right)R\phi_x(0) \\
& + \left(-U_{26}e_{21}^{(1)}\sin\omega_1 t - U_{36}e_{21}^{(2)}\sin\omega_2 t - U_{46}e_{21}^{(3)}\sin\omega_3 t\right)R\phi_y(0) \\
& + \left(-U_{27}e_{21}^{(1)}\sin\omega_1 t - U_{37}e_{21}^{(2)}\sin\omega_2 t - U_{47}e_{21}^{(3)}\sin\omega_3 t\right)R\delta\theta_z(0) \\
& + \Big[\left(U_{26}D_{6x} + U_{27}D_{7x}\right)Re_{21}^{(1)}\sin\omega_1 t + (U_{34}D_{4x} + U_{36}D_{6x}R \\
& + U_{37}D_{7x}R)e_{21}^{(2)}\sin\omega_2 t + (U_{44}D_{4x} + U_{46}D_{6x}R \\
& + U_{47}D_{7x}R)e_{21}^{(3)}\sin\omega_3 t\Big]u_x + \Big[-(U_{52}D_{2y} + U_{55}D_{5y}R)e_{21}^{(1)} \\
& \times\cos\omega_1 t - (U_{62}D_{2y} + U_{63}D_{3y} + U_{65}D_{5y}R)e_{21}^{(2)}\cos\omega_2 t \\
& - (U_{72}D_{2y} + U_{73}D_{3y} + U_{75}D_{5y}R)e_{21}^{(3)}\cos\omega_3 t + D_{2y}\Big]u_y \\
& + \Big[-(U_{52}D_{2z} + U_{55}D_{5z}R)e_{21}^{(1)}\cos\omega_1 t - (U_{62}D_{2z} + U_{63}D_{3z} \\
& + U_{65}D_{5z}R)e_{21}^{(2)}\cos\omega_2 t - (U_{72}D_{2z} + U_{73}D_{3z} \\
& + U_{75}D_{5z}R)e_{21}^{(3)}\cos\omega_3 t + D_{2z}\Big]u_z
\end{aligned}
\tag{20}
$$

More compactly, Eq. (20) can be written as

$$
\begin{aligned}
\delta y = {} & q_1(t)\delta\theta_y(0) + q_2(t)\delta\theta_x(0) + q_3(t)\delta V_x(0) \\
& + q_4(t)\delta V_y(0) + q_5(t)\phi_x(0) + q_6(t)\phi_y(0) \\
& + q_7(t)\delta\theta_z(0) + q_x(t)u_x + q_y(t)u_y + q_z(t)u_z
\end{aligned}
\tag{21}
$$

where the time-dependent coefficients q_i are identifiable by a comparison to Eq. (20).

Usage of Solution in an AQI

An AQI should be able to predict if a guided weapon can acquire a distant terrain scene by comparing, at each update time during transfer alignment, the crosstrack error calculated from the AQI algorithm against that needed for successful acquisition of the terrain scene, assuming that the weapon will fly some known period of time to a point where the acquisition is to take place. When, at some time during alignment, the AQI indicates that the crosstrack error is sufficient to permit acquisition of the high-quality terrain scene, the alignment can be terminated and the weapon subsequently launched. An AQI can also be employed in the case of GPS-only missions to predict successful acquisition of GPS.

For acquisition of a terrain scene in an unaided flyout mission, the quantity that is ultimately required is the uncertainty in the crosstrack error, that is, σ_c, at the location of the terrain scene. For the easterly (or westerly) flyout under consideration, this quantity is the square root of the variance of δy, evaluated at $t = t_f$, where the flyout time t_f is determinable by $t_f = D_{\mathrm{TER}}/V_x$. The parameter D_{TER} in this relationship is the distance of the terrain scene from the launch point. From Eq. (21), the variance of δy, evaluated at $t = t_f$, is given by

$$
\begin{aligned}
\sigma_c^2 = E(\delta y^2) = {} & q_1^2(t_f)E\left[\delta\theta_y^2(0)\right] + q_2^2(t_f)E\left[\delta\theta_x^2(0)\right] \\
& + q_3^2(t_f)E\left[\delta V_x^2(0)\right] + q_4^2(t_f)E\left[\delta V_y^2(0)\right] + q_5^2(t_f)E\left[\phi_x^2(0)\right] \\
& + q_6^2(t_f)E\left[\phi_y^2(0)\right] + q_7^2(t_f)E\left[\delta\theta_z^2(0)\right] + q_x^2(t_f)E\left(u_x^2\right) \\
& + q_y^2(t_f)E\left(u_y^2\right) + q_z^2(t_f)E\left(u_z^2\right) \\
& + \text{all possible cross correlations}
\end{aligned}
\tag{22}
$$

All of the expected values appearing here are found in the weapon Kalman filter covariance matrix. These expected values, which are calculated at each update time during the transfer alignment process, can be inserted into Eq. (22) to compute the predicted crosstrack uncertainty σ_c that would result at the site of the terrain scene if the guided weapon were to be launched at that particular update time in the transfer alignment process.

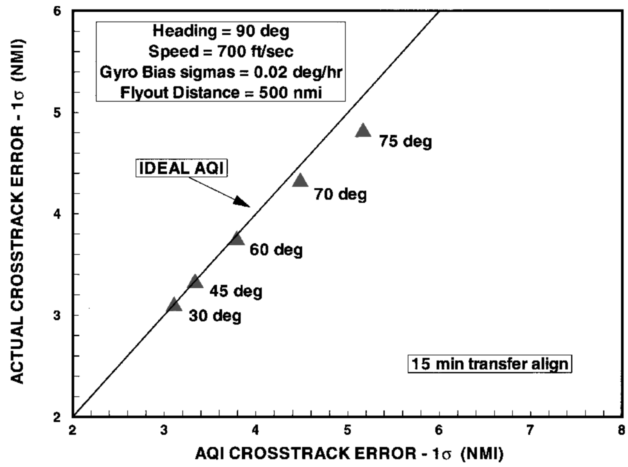

Fig. 2 Actual vs AQI crosstrack error for different latitudes.

To demonstrate that the AQI given by Eq. (22) is an acceptable AQI, Fig. 2 has been constructed. In Fig. 2, the actual crosstrack error uncertainty is plotted against the crosstrack error uncertainty predicted by Eq. (22) for various weapon launch latitudes. In the covariance simulation used to generate these results, the guided weapon receives positional updates to the weapon Kalman filter during transfer alignment every 2 s. The actual crosstrack error uncertainty differs from that predicted by the AQI in that the actual crosstrack error uncertainty includes the effects of process noise added to the Kalman filter and the dynamics of the weapon launch. Nevertheless, an examination of Fig. 2 shows the agreement to be quite satisfactory, that is, the AQI predictions are close to that of an ideal AQI. The agreement also demonstrates that the input gyro errors are dominating the entire flight. If the guided weapon is not heading nearly east or west, the AQI can still be usefully employed by using the worst-case crosstrack error uncertainty as the (conservative) indicator of when to terminate transfer alignment.

Conclusions

A closed-form solution to the seventh-order system of coupled error differential equations, appropriate for unaided flyout of a guided weapon, has been deduced. In particular, the solution can be employed to predict such important parameters as the crosstrack and downtrack errors as functions of time up to the weapon's first navigational update. To effect a closed-form solution, the heading direction of the weapon during the unaided flyout is assumed to be either east or west. The other restriction is that the equations should not be employed above 70° latitude because the variation in wander angle, taken to be zero herein, is not negligible in this case.

When treated in a probabilistic manner, it is demonstrated that these equations can be utilized to predict the standard deviations in crosstrack and downtrack errors. Such prediction capability allows these equations to be used as an AQI. This algorithm permits one to decide when to terminate the transfer alignment process of a guided weapon by predicting, at selected times in the alignment process, the crosstrack and/or downtrack error uncertainties that would result after flyout to the area where the first navigational update is expected. Usage of a covariance simulation has shown that the AQI predicted crosstrack error uncertainty at the end of transfer alignment agrees satisfactorily with the actual crosstrack error uncertainty at the end of unaided flyout, as deduced from the system covariance matrix.

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Payload Mass Fractions for Minimum-Time Trajectories of Flat and Compound Solar Sails

Colin R. McInnes*
University of Glasgow, Glasgow, Scotland G12 8QQ, United Kingdom

I. Introduction

ALMOST all solar sail mission studies documented in the published literature have considered conventional flat solar sail configurations. There is, however, an alternative configuration, which appears to offer greater performance than the flat solar sail. Compound solar sails separate the functions of collecting and directing solar radiation by using a large sun-facing collector. This collector then directs light onto a small secondary directing mirror, attached to the collector by a boom. While the main collector is fixed in a sun-facing attitude, the director is rotated to control the direction of the thrust exerted by the compound sail.

It can be shown that the thrust exerted by an ideal flat solar sail is proportional to $\cos^2 u$, where u is the angle between the sun line and the sail thrust vector, the so-called cone angle. However, for an ideal compound solar sail, it can be shown that the thrust is proportional to $\cos u$ (Ref. 1). As such, the compound sail appears more efficient than conventional flat solar sails for orbit transfer. The compound solar sail concept was originally proposed in the Soviet literature[2] in the early 1970s and was subsequently reinvented by Forward.[3] Although the concept appears attractive in principle, as yet no detailed engineering design studies have been undertaken for this solar sail configuration. In particular, the effect of losses along the optical path has yet to be addressed.

In this Note, a minimum-time steering law will be derived for a compound solar sail, which complements existing steering laws for flat solar sails. The potential benefits of compound solar sails will be investigated by calculating the increase in payload mass delivered by a compound solar sail for a simple Earth–Mars transfer problem.

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*Professor, Department of Aerospace Engineering; colinmc@aero.gla.ac.uk.

II. Minimum-Time Trajectories

To assess the performance of the compound solar sail, a steering law for minimum-time trajectories will be determined using the Pontryagin principle. Many authors have investigated minimum-time trajectories for flat solar sails, although here the succinct analysis of Ref. 4 is followed. The equations of motion for a solar sail in heliocentric orbit will be defined using nondimensional units as

$$\dot{x}_1 = x_2 \tag{1a}$$

$$\dot{x}_2 = x_3^2/x_1 - 1/x_1^2 + \beta\left(1/x_1^2\right)\cos^m u \cos u, \qquad m = 1, 2 \tag{1b}$$

$$\dot{x}_3 = -(x_2 x_3/x_1) + \beta\left(1/x_1^2\right)\cos^m u \sin u, \qquad m = 1, 2 \tag{1c}$$

$$\dot{x}_4 = x_3/x_1 \tag{1d}$$

where x_1 is the solar sail orbit radius, x_2 is the radial velocity, x_3 is the circumferential velocity, and x_4 is the polar angle of the solar sail from some reference direction. Here, the orbit radius is nondimensional with respect to the initial orbit radius whereas the velocities are nondimensional with respect to the initial circular orbit speed. With these nondimensional units, β becomes the ratio of the solar radiation pressure acceleration to the solar gravitational acceleration experienced by the solar sail. This parameter is often termed the sail lightness number.[1] The index m is used to define the type of solar sail, with $m = 2$ corresponding to a flat solar sail and $m = 1$ corresponding to a compound solar sail.

The optimal control problem is now to minimize the trip time and, hence, the following performance index:

$$J = t_f \tag{2}$$

When the Pontryagin principle is used, the Hamiltonian of the problem can be defined using costates (λ_1, λ_2, and λ_3) for each of the three states (x_1, x_2, and x_3). When these costates are used, it can be shown that

$$H = \lambda_1 x_2 + \lambda_2\left[x_3^2/x_1 - 1/x_1^2 + \beta\left(1/x_1^2\right)\cos^m u \cos u\right] + \lambda_3\left[-x_2 x_3/x_1 + \beta\left(1/x_1^2\right)\cos^m u \sin u\right] \tag{3}$$

The optimal control is now found by minimizing the Hamiltonian with respect to the control variable u. From Eq. (3), it is found that

$$\frac{\partial H}{\partial u} = \beta\frac{1}{x_1^2}\left[-\lambda_2(m+1)\cos^m u \sin u + \lambda_3(\cos^{m+1} u - m\cos^{m-1} u \sin^2 u)\right] \tag{4}$$

Therefore, it can be seen that the Hamiltonian is now stationary with respect to the control variable u under the following conditions:

$$\cos u = 0 \tag{5a}$$

$$m\lambda_3 \tan^2 u + \lambda_2(m+1)\tan u - \lambda_3 = 0 \tag{5b}$$

Furthermore, using Eq. (5b), it can be shown that the Hamiltonian is minimized for a flat solar sail and compound solar sail using the following controls for the optimum solar sail cone angle $\bar{u}$; namely,

$$\tan\bar{u} = \left(-3\lambda_2 - \sqrt{9\lambda_2^2 + 8\lambda_3^2}\right)\Big/4\lambda_3, \qquad m = 2 \tag{6a}$$

$$\tan\bar{u} = \left(-\lambda_2 - \sqrt{\lambda_2^2 + \lambda_3^2}\right)\Big/\lambda_3, \qquad m = 1 \tag{6b}$$

It is found that Eq. (6a) is identical to the steering law derived in other studies of minimum-time trajectories for flat solar sails, as